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Numerical Approximation of Nonlinear Dispersive-Diffusive Traffic Flow Model with Source Term

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Abstract

The study of traffic flow forecasting provides critical information for the planning and operational management of modern transportation systems. This paper uses a nonlinear Partial Differential Equation (PDE) with source term describing traffic phenomena to present a traffic modelling framework for traffic density evolution. The PDE is based on the nonhomogeneous KdV–Burgers equation including convection, diffusion, dispersion and source term. The governing PDE is solved by employing Finite Difference Method (FDM) under two initial conditions: with and without source term. Stability error analysis is carried out to ensure the convergence of the method. The numerical results computed by using MATLAB together with solution graphs of traffic density waves show the influence of source term on the solution of PDE.

Keywords: Traffic Flow, Differential Equations, Finite Difference Method, Mathematical Modeling, Density, Velocity, LWR Model.

1. Introduction

One of the major problems with transportation system in densely populated urban areas is traffic congestion. The main causes of traffic congestion are limited road capacity, large number of vehicles, poor traffic administration. Consequences of traffic congestion such as, longer traffic times, increased vehicle emissions, fuel watage, etc, lead to environmental pollution and loss of economy (Daganzo, 1994; Wagner et al., 2009). So, the study traffic flow evolution and behaviour is significant for developing efficient transportation systems which enable us to enhance efficient congestion management system (Laval, 2023).

The flow of traffic congestion can be modelled by using nonlinear Partial Differential Equations (PDEs) describing the traffic density. These mathematical models enable us to forecast traffic dynamics using continuum approaches (Whitham, 1974).

The major drawback of the classical Lighthill–Whitham–Richards (LWR) is that it only describes traffic density evolution through nonlinear conservation laws without incorporating diffusion, dispersion, and external disturbances (Lighthill & Whitham, 1955; Richards, 1956). So, it is necessary to develop higher-order models incorporating diffusion and dispersion (Aw & Rascle, 2000; Ge et al., 2012b; Helbing, 2001; Zhang, 2002).

A nonlinear dispersive-dissipative traffic model with external forcing is considered in this paper to analyse traffic dynamics. This model is based on KdV-Burgers equation and is defined over a bounded spatial domain $\Omega = [a, b] \subset \mathbb{R}$ and a finite time interval $(0, T]$ given by

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}=\nu\frac{\partial^2 u}{\partial x^2}-\beta\frac{\partial^3 u}{\partial x^3}+S(x,t) \tag{1}$$

Where u refers to traffic density which is the function of space x and time t. The diffusion coefficient ν refers to smoothing effects due to driver behaviour and the dispersion coefficient β is the responsible for wave profile(Zhai et al., 2023), (Coclite et al., 2024; Kerner, 2004; Ge et al., 2012a)**.**The source term denoted by $S(x,t)$ represents the external effects such as inflow/ outflow, traffic signals, and lane-changing effects (Sreekumara et al., 2021; Thankappan et al., 2015).

The nonlinear convective term $u\frac{\partial u}{\partial x}$ arises from fundamental conservation principles of traffic flow, while the diffusion term $\nu\frac{\partial^2 u}{\partial x^2}$ is introduced to model relaxation and viscous effects in second-order traffic models. The dispersive term $\beta\frac{\partial^3 u}{\partial x^3}$ is motivated by nonlinear wave theory and has been shown to play a significant role in capturing oscillatory traffic patterns and self-sustained nonlinear waves, providing a more realistic representation of traffic dynamics compared to purely hyperbolic or parabolic models (Cheng et al., 2017; Flynn et al., 2009; Seibold et al., 2013; Tian & Gao, 2018).

Recent studies have emphasized the importance of nonlinear dispersive-diffusive models in accurately describing traffic dynamics (Lai et al., 2013). For instance, second-order and higher-order continuum models have shown improved capability in reproducing traffic shock structures and rarefaction waves (Fadhloun et al., 2014). Moreover, dispersive corrections have been found to play a crucial role in modeling phantom traffic jams and nonlinear wave interactions (Morton & Mayers, 2005; Richtmyer & Morton, 1967). Numerical methods such as finite difference methods (FDM) and lattice-based discretizations have been widely used to solve such equations due to their simplicity and computational efficiency (Smith, 1985; Hirsch, 1990; Fletcher, 1991; Ames, 1992; Thomas, 1995; LeVeque, 2002).

This paper deals with solving a nonlinear traffic flow model describing convection, diffusion, dispersion, and source effects by using FDM which enhances the efficiency of numerical traffic flow models and their applicability to real-world scenarios.

The next part of the paper is arranged as under: In Section 2, the numerical implementation of FDM is described, Section 3 contains results and discussion, in Section 4 the conclusion is given, and Section 5 includes references.

2. Numerical Implementation of Finite Difference Method (FDM)

Domain of (1) : $x\in[a,b], t>0$

Initial Condition(IC): $u(x,0) = u_0(x) = e^{-\left(\frac{(x-x_0)^2}{\sigma^2}\right)}$

Boundary conditions (BCs): $u(a,t)=0, u(b,t)=0 \; for\; all\; t>0$

$$x_i = a+i\Delta x (i=0,1,2,\ldots N), t^n = n\Delta t,\; u(x_i,t^n)={u_i}^n \tag{2}$$

The time derivative by forward difference formula is given by

$$\frac{\partial u}{\partial t}=u_t\approx\frac{{u_i}^{n+1}-{u_i}^n}{\Delta t} \tag{3}$$

The nonlinear convection term by central difference is given by

$$u\frac{\partial u}{\partial x}=\frac{1}{2}\left(\frac{\partial(u^2)}{\partial x}\right)=\left(\frac{(u_{i+1})^2-(u_{i-1})^2}{4\Delta x}\right) \tag{4}$$

The first, second and third derivatives are given by

$$\frac{\partial u}{\partial x} = u_x \approx \frac{{u_{i+1}}^n - {u_{i-1}}^n}{2\Delta x} \tag{5}$$

$$\frac{\partial^2 u}{\partial x^2} = u_{xx} \approx \frac{{u_{i+1}}^n - 2{u_i}^n + {u_{i-1}}^n}{(\Delta x)^2} \tag{6}$$

$$\frac{\partial^3 u}{\partial x^3} = u_{xxx} \approx \frac{{u_{i+2}}^n - 2{u_{i+1}}^n + 2{u_{i-1}}^n - {u_{i-2}}^n}{2(\Delta x)^3} \tag{7}$$

$${u_0}^n = 0, {u_N}^n = 0$$

Solving by using from (1) to (7) for ${u_i}^{n+1}$ to get the iterative FDM solution as below:

$${u_i}^{n+1} = {u_i}^n - \Delta t\left[\frac{(u_{i+1})^2 - (u_{i-1})^2}{4\Delta x}\right] + \nu\Delta t\left[\frac{{u_{i+1}}^n - 2{u_i}^n + {u_{i-1}}^n}{(\Delta x)^2}\right] - \beta\Delta t\left[\frac{{u_{i+2}}^n - 2{u_{i+1}}^n + 2{u_{i-1}}^n - {u_{i-2}}^n}{2(\Delta x)^3}\right] + \Delta t {S_i}^n \tag{8}$$

3. Results and Discussion

The numerical solution is visualized using 3D surface and contour plots, illustrating the evolution of traffic density over space and time.

Let us calculate first few iterations by choosing the following:

3.1. When $S(x,t) = 0$

Domain: $[0,1]$, Grid points: $N = 5 \Rightarrow \Delta x = 0.2$, Time step: $\Delta t = 0.01, \nu = 0.5, \beta = 0.01, \sigma = 1$, and IC: $u(x,0) = e^{-(x-0.5)^2}$

Grid: $x_i = 0{,}0.2{,}0.4{,}0.6{,}0.8{,}1.0$

The stability conditions for $= 0.5, \beta = 0.01$,

$$\nu\frac{\Delta t}{(\Delta x)^2} = 0.5\frac{0.01}{(0.2)^2} = 0.125 \leq 0.5 \tag{9}$$

$$\beta\frac{\Delta t}{2(\Delta x)^3} = 0.01\frac{0.01}{2\times(0.2)^3} = 0.00635 \leq 0.25 \tag{10}$$

So, both the stability conditions are satisfied.

Using FDM formula by (8), and the initial values, the updated values of the solution are obtained and the computed results are presented in Table 1 and the same are depicted in Figure 1.

Table 1: Traffic density evolution of (1) when *S(x,t)=0*

i	x_i	${u_i}^0 = e^{-(x-0.5)^2}$	${u_i}^1$	${u_i}^2$
0	0.0	0.0000	0.0000	0.0000
1	0.2	0.9139	0.7827	0.6913
2	0.4	0.9900	0.9835	0.9561

3	0.6	0.9900	0.9776	0.9609
4	0.8	0.9139	0.8357	0.7719
5	1.0	0.0000	0.0000	0.0000

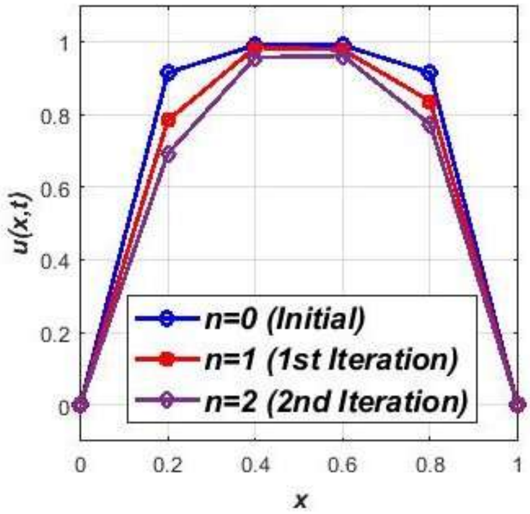

Figure 1: Plot of Numerical solution of (1) for *S(x, t)=0* by using FDM

The numerical results also describe the evolution of traffic density under the combined influence of nonlinear convection, diffusion, and dispersion. The initial condition represents a symmetric Gaussian profile centred at $x_0 = 0.5$ over the domain $[0,1]$. The values at the boundary nodes $i = 0$ and $i = 5$ are set to zero as per the Dirichlet BCs.The profile is perfectly symmetric about $x = 0.5$, satisfying ${u_i}^0 = {u_{N-i}}^0$ which is consistent due to the even symmetry of the Gaussian function. So, the initial traffic density is heavily concentrated near the centre of the domain and vanishes at the boundaries, representing a localized congestion zone at $t = 0$.

As the iteration progresses from $n = 0$ to $n = 1$ and $n = 2$, the density values at $i = 1$ and $i = 4$ decrease progressively, indicating that vehicles near the boundary region are dispersing outward and the congestion is being smoothed by the strong diffusive effect of $\nu = 0.5$. Also, at $i = 2$ and $i = 3$ show a gradual decline, indicating that the overall congestion peak is being dissipated over time. The asymmetry at $i = 2$ and $i = 3$ across iterations indicates the influence of the nonlinear convection term, which drives the traffic density wave to propagate in the direction of flow. It is shown by Table 1 that The PDE model presented in the paper captures the realistic traffic phenomena.

Table 2 presents the relative % change (error) between two iterations by using the following formula. It shows the decrease in the % changes verifying the convergence of the method.

$$e_i^n = \frac{|{u_i}^{n+1} - {u_i}^n|}{|{u_i}^n|} \times 100\% \qquad (11)$$

Table 2: Relative % changes between two iterations of FDM

i	x_i	${e_i}^{0\to1}$ %	${e_i}^{1\to2}$ %
1	0.2	14.36%	11.68%
2	0.4	0.66%	2.79%
3	0.6	1.25%	1.71%
4	0.8	8.56%	7.63%

3.2. For $S(x,t) = sin(\pi x)e^{-t}$.

Domain: $[0, 1]$, Grid points: $N = 5 \Rightarrow \Delta x = 0.2$, Time step: $\Delta t = 0.01, \nu = 0.5,$

$\beta = 0.01, \sigma = 1,$ and IC: $u(x, 0) = e^{-(x-0.5)^2}$ Grid: $x_i = 0,0.2,0.4,0.6,0.8,1.0$

Using the same FDM formula by (8), and the initial values, the updated values of the solution are obtained and the computed traffic density values at each grid nodes and different time levels are tabulated in Table 3 and the same are depicted in Figure 2.

Table 3 : Traffic density evolution of (1) when $\mathrm{S(x,t) = sin(\pi x)e^{-t}}$

i	x_i	${u_i}^0 = e^{-(x-0.5)^2}$	${u_i}^1$	${u_i}^2$
0	0.0	0.0000	0.0000	0.0000
1	0.2	0.9139	0.7885	0.7025
2	0.4	0.9900	0.9930	0.9745
3	0.6	0.9900	0.9871	0.9795
4	0.8	0.9139	0.8416	0.7835
5	1.0	0.0000	0.0000	0.0000

It is observed from Table 3 that the initial Gaussian density profile is symmetric about the centre $x_0 = 0.5$, with boundary nodes fixed at zero and peak values of 0.9900 at $i = 2$ and $i = 3$.

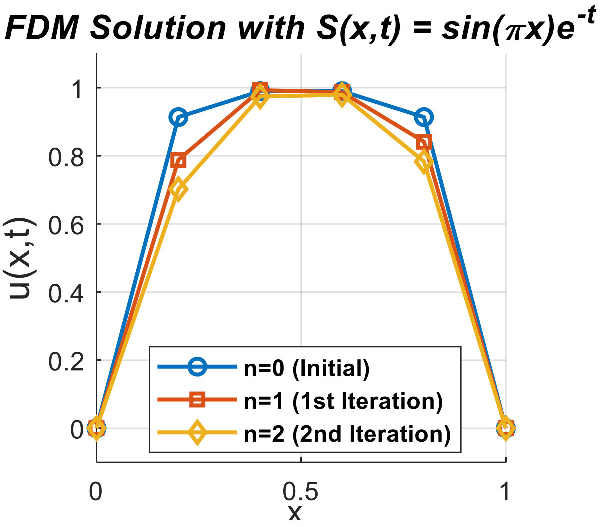

Figure 2: Graph of Numerical solution of (1) for $S(x,t) = sin(\pi x)e^{-t}$ by using FDM

Figure 2 depicts the surface plot of the traffic density evolution over space and time for $S(x,t) = sin(\pi x)e^{-t}$. By adding the source term it is possible to depict a more realistic traffic wave pattern.

The comparison of solutions of (1) for $S(x,t) = 0$ and $S(x,t) = sin(\pi x)e^{-t}$ is presented in Figure 3.

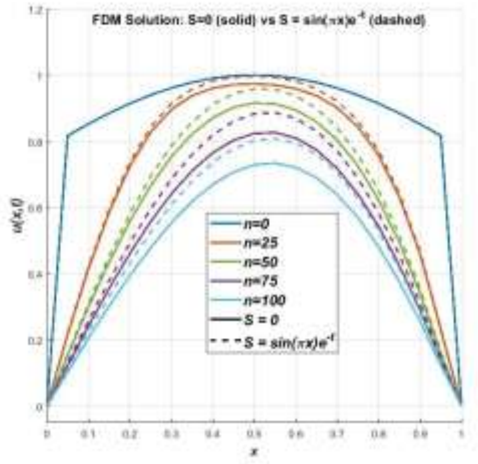

Figure 3: Comparison of solution graphs of (1) with $S(x,t) = 0$ and $S(x,t) = sin(\pi x)e^{-t}$ by using FDM

At $n = 0$, both the solutions exhibit identical Gaussian density profile. As the time increases, it shows the density profile gradually flattening and dissipating outward. But when source term is added, the solution maintains consistently higher density values at all-time levels. It is observed from the Figure 3 that while the solid lines approach almost-zero density, the dashed lines retain a significantly elevated profile. This indicates the impact of source term on traffic density.

4. Conclusion

In this paper, a nonlinear dispersive-diffusive traffic flow model with source term, has been numerically solved by using FDM. The PDE model is solved by considering $S(x,t) = 0$ and $S(x,t) = sin(\pi x)e^{-t}$.

Errors have been calculated to ensure the convergence of the method. A comparative analysis between PDE with source term and without source term has been made to understand the influence of source term on the solution of PDE. So, it is concluded that the FDM provides a stable, consistent and convergent computational technique for analyzing nonlinear traffic flow dynamics. And also the present model can be applied to study traffic flow evolution and behaviour which help us in developing efficient transportation systems.

5. References

1. Ames, W. F. (1992). *Numerical methods for partial differential equations*. Academic Press.
2. Aw, A., & Rascle, M. (2000). Resurrection of "second order" models of traffic flow. *SIAM Journal on Applied Mathematics*, 60(3), 916–938.
3. Cheng, R., Ge, H., & Wang, J. (2017). KdV–Burgers equation in a new continuum model based on full velocity difference model considering anticipation effect. *Physica A: Statistical Mechanics and its Applications*, 481, 52–59.
4. Coclite, G. M., Karlsen, K. H., & Risebro, N. H. (2024). Well-posedness of a traffic flow model with phase transitions. *Zeitschrift für angewandte Mathematik und Physik*, 75, Article 21.
5. Daganzo, C. F. (1994). The cell transmission model: A dynamic representation of highway traffic. *Transportation Research Part B*, 28(4), 269–287.
6. Fadhloun, K., Rakha, H., & Loulizi, A. (2014). Framework for estimating traffic flow rates past moving bottlenecks. *Transportation Research Record*, 2422(1), 59–67.
7. Fletcher, C. A. J. (1991). *Computational techniques for fluid dynamics*. Springer.
8. Flynn, M. R., Kasimov, A. R., Nave, J. C., Rosales, R. R., & Seibold, B. (2009). Self-sustained nonlinear waves in traffic flow. *Physical Review E*, 79(5), 056113.
9. Ge, H. X., Dai, S. Q., & Dong, L. Y. (2012a). A study of wide moving jams in traffic flow with anticipation effects. *Physica A: Statistical Mechanics and its Applications*, 391(23), 5971–5977.
10. Ge, H. X., & Lo, S. M. (2012b). The KdV–Burgers equation in speed gradient viscous continuum model. *Physica A: Statistical Mechanics and its Applications*, *391*(4), 1652–1656.
11. Helbing, D. (2001). Traffic and related self-driven many-particle systems. *Reviews of Modern Physics*, 73(4), 1067–1141.
12. Hirsch, C. (1990). *Numerical computation of internal and external flows*. Wiley.
13. Kerner, B. S. (2004). *The physics of traffic*. Springer.
14. Lai, L. L., Cheng, R. J., Li, Z. P., & Ge, H. X. (2013). The KdV–Burgers equation in a modified speed gradient continuum model. *Chinese Physics B*, 22(6), 060511.
15. Laval, J. A. (2023). Self-organized criticality of traffic flow: Implications for congestion management technologies. *Transportation Research Part C: Emerging Technologies*, 149, 104056.
16. LeVeque, R. J. (2002). *Finite volume methods for hyperbolic problems*. Cambridge University Press.
17. Lighthill, M. J., & Whitham, G. B. (1955). On kinematic waves II: A theory of traffic flow on long crowded roads. *Proceedings of the Royal Society A*, 229(1178), 317–345.
18. Morton, K. W., & Mayers, D. F. (2005). *Numerical solution of partial differential equations*. Cambridge University Press.
19. Richards, P. I. (1956). Shock waves on the highway. *Operations Research*, 4(1), 42–51.
20. Richtmyer, R. D., & Morton, K. W. (1967). *Difference methods for initial-value problems*. Wiley.
21. Seibold, B., Flynn, M. R., Kasimov, A. R., & Rosales, R. R. (2013). Constructing set-valued fundamental diagrams from jamiton solutions in second order traffic models. *Networks and Heterogeneous Media*, *8*(3), 745–772.

22. Smith, G. D. (1985). *Numerical solution of partial differential equations: Finite difference methods*. Oxford University Press.
23. Sreekumara, M., Joshi, S. M., Mathew, T. V., & Chatterjee, A. (2021). A multi-class traffic flow model for disordered traffic behavior. *Transportmetrica B: Transport Dynamics*, 10(1), 48–67.
24. Thankappan, A., Sunny, A., Vanajakshi, L., & Subramanian, S. C. (2015). A non-continuum lumped-parameter dynamic model applied to Indian traffic. *Systems Science & Control Engineering*, 3(1), 320–331.
25. Thomas, J. W. (1995). *Numerical partial differential equations*. Springer.
26. Tian, F., & Gao, Z. (2018). Travelling wave solutions of perturbed mKdV equation representing traffic congestion. *Wave Motion*, 79, 57–72.
27. Wagner, P., Brockfeld, E., Gartner, N. H., & Sohr, A. (2009). Fundamental diagram of traffic flow on urban roads. *Transportation Research Record*, 2124(1), 183–191.
28. Whitham, G. B. (1974). *Linear and nonlinear waves*. Wiley.
29. Zhai, C., Wu, W., & Xiao, Y. (2023). Modeling continuous traffic flow with the average velocity effect of multiple vehicles ahead on gyroidal roads. *Digital Transportation and Safety*, *2*(2), 124–138.
30. Zhang, H. M. (2002). A non-equilibrium traffic model devoid of gas-like behavior. *Transportation Research Part B: Methodological*, 36(3), 275–290.