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Introduction to the Almost Suslin Matrices-The Key Lemma

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Abstract

The Suslin matrices exhibit interesting properties which lead to the Local Global Principle and other important results. In this paper, we follow the Suslin construction by defining a variant of the Suslin matrices which are named the Almost Suslin Matrices $S_r^{\lambda,\mu}(v,w)$, w.r.t. a pair of parameters $\lambda,\mu \in R$. The Almost Suslin matrices, despite the parameters, satisfy properties which can be used to derive stronger results. We define a special product $\langle\langle v,w\rangle\rangle$ for $v,w \in M_{1(r+1)}(R)$. This helps us to derive results which finally leads us to our goal of proving the Key Lemma.

Key Words: Almost Suslin matrices, parameters, Key Lemma

1. Introduction

In his paper [9], A.A. Suslin, inspired by the Vaserstein symbol, constructed a special type of matrices which satisfy interesting properties. It was this symbol which led him to construct these matrices. In ([10], Proposition 1.6), he proved that a unimodular row of the form $(a_0, a_1, a_2^2, \dots, a_r^r)$ can be completed to an invertible matrix. Given rows $v, w \in \mathrm{Um}_{r+1}(R)$, with $\langle v, w\rangle = 1$, Suslin described an inductive procedure in ([9], §5) for constructing the matrix $S_r(v,w) \in SL_{2^r}(R)$, whose size can be reduced by elementary row and column operations to

$r+1$ to get a matrix of determinant 1, which is a completion of $(a_0, a_1, a_2^2, \dots, a_r^r)$. We call $S_r(v,w)$ the Suslin matrix w.r.t. the pair (v,w).

Selby Jose–Rao in [3], [4] studied the Suslin matrices and obtained the Key Lemma as well as the Fundamental property of Suslin matrices which enabled them to develop the Jose–Rao theory, attaching Reflections to a Suslin matrix.

In this article we build up on the Suslin construction by defining Suslin Matrices with parameters $S_r^{\lambda,\mu}(v,w)$, w.r.t. a pair of parameters $\lambda,\mu \in R$, satisfying the above equation. When $\lambda = 1 = \mu$ we recover the Suslin matrices. We name these matrices Almost Suslin Matrices.

2. Preliminaries

In this section, we revisit the concept of Suslin matrices along with their properties. This motivates us to prove some properties of Almost Suslin Matrices, which become a useful tool for the results in subsequent work. Suslin matrices, $S_r(v,w)$ of size 2^r were defined inductively in [9]. Let $v = (a_0, a_1, \dots, a_r) = (a_0, v_1)$, $w = (b_0, b_1, \dots, b_r) = (b_0, w_1) \in M_{1(r+1)}(R)$, where $v_1, w_1 \in M_{1r}(R)$. Set $S_0(v,w) = a_0$, and

$$S_r(v,w) = \begin{pmatrix} a_0 I_{2^{r-1}} & S_{r-1}(v_1,w_1) \\ -S_{r-1}(w_1,v_1)^T & b_0 I_{2^{r-1}} \end{pmatrix}$$

where the superscript T stands for the transpose of a matrix.

The following properties were noted in ([9], Lemma 5.1):

Lemma 2.1. *The Suslin matrices satisfy the following properties: Let $\langle v,w\rangle = vw^T$. Then*

1. $S_r(v,w)S_r(w,v)^T = \langle v,w\rangle I_{2^r} = S_r(w,v)^T S_r(v,w)$.
2. $\det S_r(v,w) = \langle v,w\rangle^{2^{r-1}}$.

Lemma 2.2. **([3], Lemma 3.1)** *Let v, w, s, $t \in M_{1(r+1)}(R)$ and let $v = (a_0, a_1, \dots, a_r)$,*

$w = (b_0, b_1, \dots, b_r)$. Then

$$\begin{aligned} S_r(v,w) + S_r(w,v)^T &= (a_0 + b_0) I_{2^r}. \\ S_r(s,t)S_r(w,v)^T + S_r(v,w)S_r(t,s)^T &= \{\langle s,w\rangle + \langle v,t\rangle\} I_{2^r}. \\ S_r(w,v)^T S_r(s,t) + S_r(t,s)^T S_r(v,w) &= \{\langle s,w\rangle + \langle v,t\rangle\} I_{2^r}. \end{aligned}$$

In future, we need the following well-known property of determinants.

Lemma 2.3. **([9], Lemma 1.3)** *Let R be a commutative ring. Let $A, B, C, D \in M_r(R)$ with $AB = BA$. Then,* $\det\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \det(DA - CB)$.

In [9], A.A. Suslin also defines a sequence of forms $J_r \in M_{2^r}(R)$, by the recurrence formulae as follows:

Definition 2.4. *For $r \geq 0$,* $J_r = \begin{cases} 1 & \text{for } r = 0 \\ J_{r-1} \perp -J_{r-1}, & \text{for } r \text{ even} \\ J_{r-1} \top -J_{r-1}, & \text{for } r \text{ odd.} \end{cases}$

(The English translation wrongly says $J_r = J_{r-1} \perp J_{r-1}$, when r is even).

Lemma 2.5. **(A. A. Suslin) ([9], Lemma 5.2)**
For $r \geq 0$, the matrix J_r has the following properties:

1. $J_r^{-1} = J_r^T = (-1)^{r(r+1)/2} J_r$.
2. $\det J_r = 1$.

Remark 2.6. *Note that by Lemma 5(i), for $r = 4k + 1$ and $r = 4k + 2$, J_r is an antisymmetric matrix, and for $r = 4k + 3$ and $r = 4k$, J_r is a symmetric matrix.*

A.A. Suslin also proved the following:

Lemma 2.7. **(Suslin Identities) ([9], Lemma 5.3)**

1. *For $r = 4k$,* $(S_r(v,w)J_r)^T = S_r(v,w)J_r$.
2. *For $r = 4k + 1$,* $S_r(v,w)J_r S_r(v,w)^T = \langle v,w\rangle J_r$.
3. *For $r = 4k + 2$,* $(S_r(v,w)J_r)^T = -S_r(v,w)J_r$.

4. *For* $r = 4k + 3$, $S_r(v,w)J_rS_r(v,w)^T = \langle v,w\rangle J_r$.

Remark 2.8. We have the following via Lemma 2.7:

$$J_rS_r(v,w)^TJ_r^{-1} = \begin{cases} S_r(v,w) & \text{if } r \text{ even} \\ S_r(w,v)^T & \text{if } r \text{ odd} \end{cases}$$

Notation 2.9. For a matrix $\alpha \in M_k(R)$, we define α^{top} as the matrix whose entries are the same as that of α above the diagonal, and on the diagonal, and is zero below the diagonal. Similarly, we define α^{bot}.

For simplicity we may write α^t for α^{top}, α^b for α^{bot} and α^T for α transpose. Moreover, we use α^{tb} for α^{top} or α^{bot}.

We also write expressions like $x\alpha^{tb}x^{-1} = \beta^{bt}$ to mean that both $x\alpha^{top}x^{-1} = \beta^{bot}$ and $x\alpha^{bot}x^{-1} = \beta^{top}$ hold.

3. Almost Suslin Matrix: Definition and Properties

We begin by describing the process by which the Almost Suslin Matrix with parameters λ, μ $S_r^{\lambda\mu}(v,w)$ is defined. The definition reminds us of Suslin Matrices but there is a major difference which will be seen below. Let R be a commutative ring with unity.

Definition 3.1. *Let* $\lambda, \mu \in R$ *and* $v = (a_0, a_1, \dots, a_r) = (a_0, v_1)$, $w = (b_0, b_1, \dots, b_r)$

$= (b_0, w_1) \in M_{1(r+1)}(R)$, *where* $v_1, w_1 \in M_{1r}(R)$. *Set*

$$S_0^{\lambda,\mu}(v,w) = a_0,$$

and

$$S_r^{\lambda,\mu}(v,w) = \begin{pmatrix} a_0 I_{2^{r-1}} & \lambda S_{r-1}(v_1,w_1) \\ -\mu S_{r-1}(w_1,v_1)^T & b_0 I_{2^{r-1}} \end{pmatrix}.$$

Note that when $\lambda = \mu = 1$, $S_r^{\lambda,\mu}(v,w)$ is a Suslin matrix.

For $v, w \in M_{1(r+1)}(R)$ as above, we define

$$\langle\langle v,w\rangle\rangle = a_0b_0 + \lambda\mu \sum_{i=1}^{r} a_i\, b_i = a_0b_0 + \langle \lambda v_1, \mu w_1\rangle = \langle (a_0, \lambda v_1), (b_0, \mu w_1)\rangle.$$

The following properties of $\langle\langle v,w\rangle\rangle$ can be proved easily.

Lemma 3.2. *Let* $v, w, s \in M_{1(r+1)}(R)$ *and* $\alpha \in R$. *Then*

1. $\langle\langle v,w\rangle\rangle = \langle\langle w,v\rangle\rangle$.
2. $\langle\langle v+w,s\rangle\rangle = \langle\langle v,s\rangle\rangle + \langle\langle w,s\rangle\rangle$.

3. $\langle\langle \alpha v, w\rangle\rangle = \alpha\langle\langle v, w\rangle\rangle$.

It is clear that the product $\langle\langle v, w\rangle\rangle$ is not an inner product.

We prove some useful properties of $S_r^{\lambda,\mu}(v, w)$.

Lemma 3.3. *Let* $\lambda, \mu \in R$, $v = (a_0, a_1, \dots, a_r) = (a_0, v_1)$, $w = (b_0, b_1, \dots, b_r) = (b_0, w_1)$ $\in M_{1(r+1)}(R)$. *Then*

1. $S_r^{\lambda,\mu}(v, w) S_r^{\mu,\lambda}(w, v)^T = \langle\langle v, w\rangle\rangle I_{2^r} = S_r^{\mu,\lambda}(w, v)^T S_r^{\lambda,\mu}(v, w)$.
2. $\det\left(S_r^{\lambda,\mu}(v, w)\right) = \langle\langle v, w\rangle\rangle^{2^{r-1}}$.
3. $\operatorname{adj} S_r^{\lambda,\mu}(v, w) = \langle\langle v, w\rangle\rangle^{2^{r-1}-1} S_r^{\mu,\lambda}(w, v)^T$.
4. $S_r^{\mu,\lambda}(w, v)^T = S_r^{\lambda,\mu}\big((b_0, -v_1), (a_0, -w_1)\big)$.

Proof: We obtain

$$S^{\lambda,\mu}(v, w) S^{\mu,\lambda}(w, v)^T = \begin{pmatrix} P & 0 \\ 0 & P \end{pmatrix}$$

where $P = a_0 b_0 I_{2^{r-1}} + \lambda\mu S_{r-1}(v_1, w_1) S_{r-1}(w_1, v_1)^T$. Since $S_{r-1}(v_1, w_1) S_{r-1}(w_1, v_1)^T = \langle v_1, w_1\rangle I_{2^{r-1}}$ (Lemma 2.1), we have

$$P = \left(a_0 b_0 + \lambda\mu \sum_{i=1}^{r} a_i\, b_i\right) I_{2^{r-1}}.$$

Therefore,

$$S^{\lambda,\mu}(v, w) S^{\mu,\lambda}(w, v)^T = \langle\langle v, w\rangle\rangle I_{2^r}$$

proving (a).

Since $(a_0 I_{2^{r-1}})\big(\lambda S_{r-1}(v_1, w_1)\big) = \big(\lambda S_{r-1}(v_1, w_1)\big)(a_0 I_{2^{r-1}})$, applying Lemma 2.3 we get

$$\begin{aligned}
\det\left(S_r^{\lambda,\mu}(v, w)\right) &= \det\big((b_0 I_{2^{r-1}})(a_0 I_{2^{r-1}}) + \lambda\mu S_{r-1}(w_1, v_1)^T S_{r-1}(v_1, w_1)\big) \\
&= \det\big(a_0 b_0 I_{2^{r-1}} + \lambda\mu S_{r-1}(w_1, v_1)^T S_{r-1}(v_1, w_1)\big) \\
&= \det(a_0 b_0 I_{2^{r-1}} + \lambda\mu \langle v_1, w_1\rangle I_{2^{r-1}}) \text{ [By Lemma 2.1]} \\
&= \det\left(a_0 b_0 + \lambda\mu \sum_{i=1}^{r} a_i\, b_i\right) I_{2^{r-1}} \\
&= \left(a_0 b_0 + \lambda\mu \sum_{i=1}^{r} a_i\, b_i\right)^{2^{r-1}} = \langle\langle v, w\rangle\rangle^{2^{r-1}}
\end{aligned}$$

and (b) is proved.

The property of determinants gives

$$S_r^{\mu,\lambda}(w,v)^T S_r^{\lambda,\mu}(v,w)\ \text{adj}\ \left(S_r^{\lambda,\mu}(v,w)\right) = \det\left(S_r^{\lambda,\mu}(v,w)\right) S_r^{\mu,\lambda}(w,v)^T.$$

By using (a) and (b) we get

$$\langle\langle v,w\rangle\rangle\ \text{adj}\ \left(S_r^{\lambda,\mu}(v,w)\right) = \langle\langle v,w\rangle\rangle^{2^{r-1}} S_r^{\mu,\lambda}(w,v)^T.$$

If $\langle\langle v,w\rangle\rangle$ is not a zero divisor, then

$$\text{adj}\ \left(S_r^{\lambda,\mu}(v,w)\right) = \langle\langle v,w\rangle\rangle^{2^{r-1}-1} S_r^{\mu,\lambda}(w,v)^T.$$

If $\langle\langle v,w\rangle\rangle$ is a zero divisor, then considering $v' = (a_0 + X, a_1, \dots, a_r)$ and

$w' = (b_0 + X, b_1, \dots, b_r) \in M_{1(r+1)}(R[X])$, $\langle\langle v',w'\rangle\rangle$ is not a zero divisor in the ring $R[X]$ as it is a unitary polynomial. So, by the case above, we get

$$\text{adj}\ S_r^{\lambda,\mu}(v',w') = \langle\langle v',w'\rangle\rangle^{2^{r-1}-1} S_r^{\mu,\lambda}(w,v)^T.$$

Setting $X = 0$ gives our desired result (c).

Property (d) follows by a direct computation. □

Lemma 3.4. *The following formulae are valid for the Almost Suslin Matrices:*

1. *For* $r = 4k$, $\left(S_r^{\lambda,\mu}(v,w)J_r\right)^T = S_r^{\mu,\lambda}(v,w)J_r$.
2. *For* $r = 4k+1$, $S_r^{\lambda,\mu}(v,w)J_r S_r^{\lambda,\mu}(v,w)^T = \langle\langle v,w\rangle\rangle J_r$.
3. *For* $r = 4k+2$, $\left(S_r^{\lambda,\mu}(v,w)J_r\right)^T = -S_r^{\mu,\lambda}(v,w)J_r$.
4. *For* $r = 4k+3$, $S_r^{\lambda,\mu}(v,w)J_r S_r^{\lambda,\mu}(v,w)^T = \langle\langle v,w\rangle\rangle J_r$.

Proof: We use induction on r. The case when $r = 0$ is trivial.

When $r = 1$, we get

$$\begin{aligned} S_1^{\lambda,\mu}(v,w)J_1 S_1^{\lambda,\mu}(v,w)^T &= \{a_0 b_0 + \lambda\mu a_1 b_1\}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \\ &= \langle\langle v,w\rangle\rangle J_1. \end{aligned}$$

For $r = 2$, we obtain,

$$-S_2^{\mu,\lambda}(v,w)J_2 = \begin{pmatrix} -a_0 J_1 & \mu S_1(v_1,w_1)J_1 \\ \lambda S_1(w_1,v_1)^T J_1 & b_0 J_1 \end{pmatrix}.$$

Also,

$$S_2^{\lambda,\mu}(v,w)J_2 = \begin{pmatrix} a_0J_1 & -\lambda S_1(v_1,w_1)J_1 \\ -\mu S_1(w_1,v_1)^TJ_1 & -b_0J_1 \end{pmatrix}.$$

So, by Lemma 2.5 and Lemma 2.7,

$$\left(S_2^{\lambda,\mu}(v,w)J_2\right)^T = \begin{pmatrix} -a_0J_1 & \mu J_1S_1(w_1,v_1) \\ \lambda J_1S_1(v_1,w_1)^T & b_0J_1 \end{pmatrix}$$

By Lemma 2.1 and Lemma 2.7 (using the case when $r = 4k + 1$) we get,

$$\begin{aligned} S_1(v_1,w_1)J_1S_1(v_1,w_1)^TS_1(w_1,v_1) &= \langle v_1,w_1\rangle J_1S_1(w_1,v_1) \\ \Rightarrow \langle v_1,w_1\rangle S_1(v_1,w_1)J_1 &= \langle v_1,w_1\rangle J_1S_1(w_1,v_1) \\ \Rightarrow S_1(v_1,w_1)J_1 &= J_1S_1(w_1,v_1) \end{aligned}$$

whenever $\langle v_1,w_1\rangle$ is a non-zero-divisor. If $\langle v_1,w_1\rangle$ is a zero divisor we consider $v_1{}' = (a_1 + X, a_2, \dots, a_r)$ and $w_1{}' = (b_1 + X, b_2, \dots, b_r) \in M_{1r}(R[X])$. Then $\langle v_1{}', w_1{}'\rangle$ is a non-zero-divisor as it is a unitary polynomial in the ring, $R[X]$. We therefore have

$S_1(v_1{}',w_1{}')J_1 = J_1S_1(w_1{}',v_1{}')$ and putting $X = 0$, we get

$$S_1(v_1,w_1)J_1 = J_1S_1(w_1,v_1).$$

Since $J_1^T = -J_1$, by taking transpose on both sides we get,

$$J_1S_1(v_1,w_1)^T = S_1(w_1,v_1)^TJ_1.$$

Therefore, by using these results we get

$$\begin{aligned} \left(S_2^{\lambda,\mu}(v,w)J_2\right)^T &= \begin{pmatrix} -a_0J_1 & \mu S_1(v_1,w_1)J_1 \\ \lambda S_1(w_1,v_1)^TJ_1 & b_0J_1 \end{pmatrix} \\ &= -S_2^{\mu,\lambda}(v,w)J_2 \end{aligned}$$

which is the desired result.

For $r = 3$, we get

$$S_3^{\lambda,\mu}(v,w)J_3S_3^{\lambda,\mu}(v,w)^T$$
$$= \begin{pmatrix} -a_0\lambda S_2(v_1,w_1)J_2 + a_0\lambda J_2S_2(v_1,w_1)^T & a_0b_0J_2 + \lambda\mu S_2(v_1,w_1)J_2S_2(w_1,v_1) \\ -\{a_0b_0J_2 + \lambda\mu S_2(w_1,v_1)^TJ_2S_2(v_1,w_1)^T\} & \mu b_0J_2S_2(w_1,v_1) - \mu b_0S_2(w_1,v_1)^TJ_2 \end{pmatrix}.$$

Note that by Lemma 2.5 and Lemma 2.7 (using the case when $r = 4k + 2$), we have

$$\begin{aligned} S_2(v_1,w_1)J_2S_2(w_1,v_1) &= -(S_2(v_1,w_1)J_2)^TS_2(w_1,v_1) \\ &= -J_2^TS_2(v_1,w_1)^TS_2(w_1,v_1) \\ &= \langle v_1,w_1\rangle J_2. \end{aligned}$$

Similarly, $S_2(w_1,v_1)^TJ_2S_2(v_1,w_1)^T = \langle v_1,w_1\rangle J_2$. Also, by Lemma 2.5 and Lemma 2.7,

$$\begin{aligned} (S_2(v_1,w_1)J_2)^T &= -S_2(v_1,w_1)J_2 \\ \Rightarrow J_2^TS_2(v_1,w_1)^T &= -S_2(v_1,w_1)J_2 \\ \Rightarrow -J_2S_2(v_1,w_1)^T &= -S_2(v_1,w_1)J_2 \end{aligned}$$

giving $J_2S_2(v_1,w_1)^T = S_2(v_1,w_1)J_2$. Similarly, $J_2S_2(w_1,v_1) = S_2(w_1,v_1)^TJ_2$.

Hence,

$$S_3^{\lambda,\mu}(v,w)J_3S_3^{\lambda,\mu}(v,w)^T = \{a_0b_0 + \lambda\mu\langle v_1,w_1\rangle\}\begin{pmatrix} 0 & J_2 \\ -J_2 & 0 \end{pmatrix} = \langle\langle v,w\rangle\rangle J_3.$$

Hence the result is true for $r = 3$.

Suppose $r = 4k$, $k \geq 1$. Since $r-1 \equiv 4k+3$, we can express $S_{r-1}(v_1,w_1)J_{r-1}$ by Lemma 2.7 as follows:

$$\begin{aligned} S_{r-1}(v_1,w_1)J_{r-1}S_{r-1}(v_1,w_1)^TS_{r-1}(w_1,v_1) &= \langle v,w\rangle J_{r-1}S_{r-1}(w_1,v_1) \\ \Rightarrow \langle v,w\rangle S_{r-1}(v_1,w_1)J_{r-1} &= \langle v,w\rangle J_{r-1}S_{r-1}(w_1,v_1) \\ \Rightarrow S_{r-1}(v_1,w_1)J_{r-1} &= J_{r-1}S_{r-1}(w_1,v_1) \end{aligned}$$

whenever $\langle v,w\rangle$ is a non-zero-divisor. In case $\langle v,w\rangle$ is a zero divisor, we consider two vectors $v_1{}', w_1{}'$ as before in the ring $M_{1r}(R[X])$ and use the same argument, we have

$S_{r-1}(v_1,w_1)J_{r-1} = J_{r-1}S_{r-1}(w_1,v_1)$ and $S_{r-1}(w_1,w_1)^TJ_{r-1} = J_{r-1}S_{r-1}(v_1,w_1)^T$. Thus we have

$$S_r^{\mu,\lambda}(v,w)J_r = \begin{pmatrix} a_0J_{r-1} & -\mu J_{r-1}S_{r-1}(w_1,v_1) \\ -\lambda J_{r-1}S_{r-1}(v_1,w_1)^T & -b_0J_{r-1} \end{pmatrix}.$$

Hence,

$$S_r^{\lambda,\mu}(v,w)J_r = \begin{pmatrix} a_0J_{r-1} & -\lambda S_{r-1}(v_1,w_1)J_{r-1} \\ -\mu S_{r-1}(w_1,v_1)^TJ_{r-1} & -b_0J_{r-1} \end{pmatrix}.$$

Therefore, by Lemma 2.5,

$$\begin{aligned} \left(S_r^{\lambda,\mu}(v,w)J_r\right)^T &= \begin{pmatrix} a_0J_{r-1} & -\mu J_{r-1}S_{r-1}(w_1,v_1) \\ -\lambda J_{r-1}S_{r-1}(v_1,w_1)^T & -b_0J_{r-1} \end{pmatrix} \\ &= S_r^{\mu,\lambda}(v,w)J_r \end{aligned}$$

Thus, the result is true for $r = 4k$.

Now suppose $r = 4k+2$, $k \geq 1$. Since $r-1 \equiv 4k+1$, we have by Lemma 2.1, Lemma 2.5 and Lemma 2.7,

$$\begin{aligned} S_{r-1}(v_1,w_1)J_{r-1}S_{r-1}(v_1,w_1)^TS_{r-1}(w_1,v_1) &= \langle v,w\rangle J_{r-1}S_{r-1}(w_1,v_1) \\ \Rightarrow \langle v,w\rangle S_{r-1}(v_1,w_1)J_{r-1} &= \langle v,w\rangle J_{r-1}S_{r-1}(w_1,v_1) \end{aligned}$$

whenever $\langle v,w\rangle$ is a non-zero-divisor. In case it is a zero divisor, we consider two vectors in the ring $M_{1r}(R[X])$ and use the same argument as before to get

$S_{r-1}(v_1,w_1)J_{r-1} = J_{r-1}S_{r-1}(w_1,v_1)$. Similarly, $S_{r-1}(w_1,v_1)^TJ_{r-1} = J_{r-1}S_{r-1}(v_1,w_1)^T$. Hence,

$$-S_r^{\mu,\lambda}(v,w)J_r = \begin{pmatrix} -a_0J_{r-1} & \mu J_{r-1}S_{r-1}(w_1,v_1) \\ \lambda J_{r-1}S_{r-1}(v_1,w_1)^T & b_0J_{r-1} \end{pmatrix}.$$

Also, by the definition,

$$
\begin{aligned}
S_r^{\lambda,\mu}(v,w)J_r &= \begin{pmatrix} a_0 I_{2^{r-1}} & \lambda S_{r-1}(v_1,w_1) \\ -\mu S_{r-1}(w_1,v_1)^T & b_0 I_{2^{r-1}} \end{pmatrix}\begin{pmatrix} J_{r-1} & 0 \\ 0 & -J_{r-1} \end{pmatrix} \\
&= \begin{pmatrix} a_0 J_{r-1} & -\lambda S_{r-1}(v_1,w_1)J_{r-1} \\ -\mu S_{r-1}(w_1,v_1)^T J_{r-1} & -b_0 J_{r-1} \end{pmatrix}.
\end{aligned}
$$

Hence, we have

$$
\begin{aligned}
\left(S_r^{\lambda,\mu}(v,w)J_r\right)^T &= \begin{pmatrix} a_0 J_{r-1}^T & -\mu(S_{r-1}(w_1,v_1)^T J_{r-1})^T \\ -\lambda(S_{r-1}(v_1,w_1)J_{r-1})^T & -b_0 J_{r-1}^T \end{pmatrix} \\
&= \begin{pmatrix} -a_0 J_{r-1} & \mu J_{r-1} S_{r-1}(w_1,v_1) \\ \lambda J_{r-1} S_{r-1}(v_1,w_1)^T & b_0 J_{r-1} \end{pmatrix} \\
&= -S_r^{\mu,\lambda}(v,w)J_r
\end{aligned}
$$

This proves (i) and (iii).

Let $r = 4k+1$, $k \geq 1$. Since $r-1 \equiv 4k$, we have by Lemma 2.7,

$$
\begin{aligned}
(S_{r-1}(v_1,w_1)J_{r-1})^T &= S_{r-1}(v_1,w_1)J_{r-1} \\
\Rightarrow J_{r-1}^T S_{r-1}(v_1,w_1)^T &= S_{r-1}(v_1,w_1)J_{r-1} \\
\Rightarrow J_{r-1} S_{r-1}(v_1,w_1)^T &= S_{r-1}(v_1,w_1)J_{r-1}.
\end{aligned}
$$

So, $J_{r-1}S_{r-1}(v_1,w_1)^T = S_{r-1}(v_1,w_1)J_{r-1}$ and similarly $S_{r-1}(w_1,v_1)^T J_{r-1} = J_{r-1}S_{r-1}(w_1,v_1)$. We now have

$$
S_r^{\lambda,\mu}(v,w)J_r S_r^{\lambda,\mu}(v,w)^T = \begin{pmatrix} A & B \\ C & D \end{pmatrix}
$$

where:

$$
\begin{aligned}
A &= -a_0\lambda S_{r-1}(v_1,w_1)J_{r-1} + a_0\lambda J_{r-1}S_{r-1}(v_1,w_1)^T = 0, \\
B &= a_0 b_0 J_{r-1} + \lambda\mu S_{r-1}(v_1,w_1)J_{r-1}S_{r-1}(w_1,v_1), \\
C &= -a_0 b_0 J_{r-1} - \lambda\mu S_{r-1}(w_1,v_1)^T J_{r-1} S_{r-1}(v_1,w_1)^T, \\
D &= b_0\mu J_{r-1}S_{r-1}(w_1,v_1) - b_0\mu S_{r-1}(w_1,v_1)^T J_{r-1} = 0.
\end{aligned}
$$

By Lemma 2.7,

$$
S_{r-1}(v_1,w_1)J_{r-1}S_{r-1}(w_1,v_1) = J_{r-1}S_{r-1}(v_1,w_1)^T S_{r-1}(w_1,v_1) = \langle v_1,w_1\rangle J_{r-1}.
$$

Similarly,

$$
S_{r-1}(w_1,v_1)^T J_{r-1} S_{r-1}(v_1,w_1)^T = \langle v_1,w_1\rangle J_{r-1}.
$$

Hence,

$$
S_r^{\lambda,\mu}(v,w)J_r S_r^{\lambda,\mu}(v,w)^T = \begin{pmatrix} 0 & \langle\langle v,w\rangle\rangle J_{r-1} \\ -\langle\langle v,w\rangle\rangle J_{r-1} & 0 \end{pmatrix} = \langle\langle v,w\rangle\rangle J_r.
$$

In the last case we consider, $r = 4k+3$, $k \geq 1$. Since $r-1 \equiv 4k+2$ we have by Lemma 2.5 and Lemma 2.7,

$$
(S_{r-1}(v_1,w_1)J_{r-1})^T = -S_{r-1}(v_1,w_1)J_{r-1}
$$

giving $J_{r-1}S_{r-1}(v_1, w_1)^T = S_{r-1}(v_1, w_1)J_{r-1}$.

Similarly,

$$J_{r-1}S_{r-1}(w_1, v_1) = S_{r-1}(w_1, v_1)^T J_{r-1}.$$

So, we have $S_r^{\lambda,\mu}(v, w)J_r S_r^{\lambda,\mu}(v, w)^T = \begin{pmatrix} P & Q \\ R & S \end{pmatrix}$

where

$$P = 0,$$

$$Q = \langle\langle v, w \rangle\rangle J_{r-1},$$

$$R = -\langle\langle v, w \rangle\rangle J_{r-1},$$

$$S = 0.$$

Thus $S_r^{\lambda,\mu}(v, w)J_r S_r^{\lambda,\mu}(v, w)^T = \begin{pmatrix} 0 & \langle\langle v, w \rangle\rangle J_{r-1} \\ -\langle\langle v, w \rangle\rangle J_{r-1} & 0 \end{pmatrix}$

$$= \langle\langle v, w \rangle\rangle J_r.$$

This proves (ii) and (iv). □

4. Almost Unimodular Vector group

Definition 4.1. The **Almost Special Unimodular Vector group** $ASUm_r^{\lambda,\mu}(R)$ *is the subgroup of* $SL_{2^r}(R)$ *generated by the Almost Suslin matrices* $S_r^{\lambda,\mu}(v, w)$*, w.r.t the pair* (v, w) *with* $\langle\langle v, w \rangle\rangle = 1$ *for some* $w \in M_{1(r+1)}(R)$.

Definition 4.2. $AEUm_r^{\lambda,\mu}(R)^*$ *denotes the subgroup of* $ASUm_r^{\lambda,\mu}(R)$ *generated by the Almost Suslin Matrices* $S_r^{\lambda,\mu}(e_1 E_{1i}(v), e_1)$, $S_r^{\lambda,\mu}(e_1, e_1 E_{1i}(v))$ *for* $2 \leq i \leq r+1$, $v \in R$*(*e_1 *remains fixed for all generators).*

Notation 4.3. For the generators of $AEUm_r^{\lambda,\mu}(R)^*$, we use the following notations: $E_r^{\lambda,\mu}(e_i)(v) = S_r^{\lambda,\mu}(e_1 E_{1i}(v), e_1) = S_r^{\lambda,\mu}(e_1 + ve_i, e_1)$

$$E_r^{\lambda,\mu}(e_i^*)(v) = S_r^{\lambda,\mu}(e_1, e_1 E_{1i}(v)) = S_r^{\lambda,\mu}(e_1, e_1 + ve_i)$$

where $2 \leq i \leq r+1$, $v \in R$.

Notation 4.4. For a generator α of $AEUm_r^{\lambda,\mu}(R)^*$, we denote $AEUm_r^{\lambda,\mu}(R)^{tb}$ as the subgroup of $E_{2^r}(R)$ generated by the elements α^{tb}.

The generators of $AEUm_r^{\lambda,\mu}(R)^{tb}$ satisfy the splitting property and its consequence which can be easily proved.

Lemma 4.5 (Splitting Property) *Let* $\alpha, \beta \in R$. *Then* ${E_r}^{\lambda,\mu}(c_i)(\alpha+\beta)^{tb} = {E_r}^{\lambda,\mu}(c_i)(\alpha)^{tb}{E_r}^{\lambda,\mu}(c_i)(\beta)^{tb}$ *for* $c_i = e_i$ *or* $c_i = {e_i}^*$, $2 \le i \le r+1$.

Corollary 4.6. *Let* $\alpha \in R$. *Then* ${E_r}^{\lambda,\mu}(c_i)(\alpha)^{{tb}^{-1}} = {E_r}^{\lambda,\mu}(c_i)(-\alpha)^{tb}$ *for* $c_i = e_i$ *or*

$c_i = {e_i}^*$, $2 \le i \le r+1$.

5. Key Lemma

The following result helps us with computations involving the Almost Suslin Matrices and also leads us to the Key Lemma.

Lemma 5.1. *Let* $v, w, s, t \in M_{1(r+1)}(R)$ *and let* $v = (a_0, a_1, \ldots, a_r) = (a_0, v_1)$,

$w = (b_0, b_1, \ldots, b_r) = (b_0, w_1)$, $\lambda, \mu \in R$, *where* $v_1 = (a_1, \ldots, a_r)$, $w_1 = (b_1, \ldots, b_r)$. *Then*

$$S_r^{\lambda,\mu}(v,w) + S_r^{\mu,\lambda}(w,v)^T = (a_0+b_0)I_{2^r}.$$
$$S_r^{\lambda,\mu}(s,t)S_r^{\mu,\lambda}(w,v)^T + S_r^{\lambda,\mu}(v,w)S_r^{\mu,\lambda}(t,s)^T = \{\langle\langle s,w\rangle\rangle + \langle\langle v,t\rangle\rangle\}I_{2^r}.$$
$$S_r^{\mu,\lambda}(w,v)^T S_r^{\lambda,\mu}(s,t) + S_r^{\mu,\lambda}(t,s)^T S_r^{\lambda,\mu}(v,w) = \{\langle\langle s,w\rangle\rangle + \langle\langle v,t\rangle\rangle\}I_{2^r}.$$

Proof: By using the definition of the Almost Suslin Matrices, we have

$$S_r^{\lambda,\mu}(v,w) + S_r^{\mu,\lambda}(w,v)^T = (a_0+b_0)I_{2^r}.$$

Also, by Lemma $3.3(a)$ and Lemma 3.2, we have

$$S_r^{\lambda,\mu}(v+s,w+t)S_r^{\mu,\lambda}(w+t,v+s)^T = \{\langle\langle v,w\rangle\rangle + \langle\langle v,t\rangle\rangle + \langle\langle s,w\rangle\rangle + \langle\langle s,t\rangle\rangle\}I_{2^r}.$$

But $S_r^{\lambda,\mu}(v+s,w+t)S_r^{\mu,\lambda}(w+t,v+s)^T$ can also be expressed as

$$\begin{aligned} &\{S_r^{\lambda,\mu}(v,w) + S_r^{\lambda,\mu}(s,t)\}\{S_r^{\mu,\lambda}(w,v)^T + S_r^{\mu,\lambda}(t,s)^T\} \\ &\quad + S_r^{\lambda,\mu}(s,t)S_r^{\mu,\lambda}(t,s)^T \\ = \; &\langle\langle v,w\rangle\rangle I_{2^r} + S_r^{\lambda,\mu}(v,w)S_r^{\mu,\lambda}(t,s)^T + S_r^{\lambda,\mu}(s,t)S_r^{\mu,\lambda}(w,v)^T \\ &\quad + \langle\langle s,t\rangle\rangle I_{2^r}. \end{aligned}$$

Equating the two expressions for $S_r^{\lambda,\mu}(v+s,w+t)S_r^{\mu,\lambda}(w+t,v+s)^T$, we get,

$$S_r^{\lambda,\mu}(v,w)S_r^{\mu,\lambda}(t,s)^T + S_r^{\lambda,\mu}(s,t)S_r^{\mu,\lambda}(w,v)^T = \{\langle\langle v,t\rangle\rangle + \langle\langle s,w\rangle\rangle\}I_{2^r}.$$

Similarly, we can prove the last assertion. □

Lemma 5.2 (Key Lemma) *Let* $v = (a_0, a_1, \ldots, a_r), w = (b_0, b_1, \ldots, b_r) \in M_{1(r+1)}(R)$. *Then, for* $r \ge 2$, $2 \le i \le r+1$, $\nu \in R$,

1. $E_r^{\lambda,\mu}(e_i)(\nu)^{top}S_r^{\lambda,\mu}(v,w)E_r^{\lambda,\mu}(e_i)(\nu)^{bot} = S_r^{\lambda,\mu}(v - \lambda\mu\nu b_{i-1}e_1 + b_0\nu e_i, w)$
2. $E_r^{\lambda,\mu}(e_i^*)(\nu)^{top}S_r^{\lambda,\mu}(v,w)E_r^{\lambda,\mu}(e_i^*)(\nu)^{bot} = S_r^{\lambda,\mu}\big(vE_{i1}(-\lambda\mu\nu), wE_{1i}(\nu)\big)$

3. $E_r^{\lambda,\mu}(e_i)(v)^{bot}S_r^{\lambda,\mu}(v,w)E_r^{\lambda,\mu}(e_i)(v)^{top} = S_r^{\lambda,\mu}\big(vE_{1i}(v), wE_{i1}(-\lambda\mu v)\big)$

4. $E_r^{\lambda,\mu}(e_i^*)(v)^{bot}S_r^{\lambda,\mu}(v,w)E_r^{\lambda,\mu}(e_i^*)(v)^{top} = S_r^{\lambda,\mu}(v, w - \lambda\mu v a_{i-1}e_1 + a_0 v e_i)$

Proof: We use direct computation to derive the result. Let $v = (a_0, a_1, \dots, a_r)$

$= (a_0, v_1)$ and $w = (b_0, b_1, \dots, b_r) = (b_0, w_1) \in M_{1(r+1)}(R)$, where $v_1 = (a_1, a_2, \dots, a_r)$ and $w_1 = (b_1, b_2, \dots, b_r) \in M_{1r}(R)$.

Since

$$E_r{}^{\lambda,\mu}(e_i)(v) = S_r^{\lambda,\mu}(e_1 + ve_i, e_1),$$

we have by Lemma 2.2,

$$E_r{}^{\lambda,\mu}(e_i)(v)^{top}S_r^{\lambda,\mu}(v,w)E_r{}^{\lambda,\mu}(e_i)(v)^{bot}$$

$$\begin{aligned}
&= \begin{pmatrix} a_0 I_{2^{r-1}} - \lambda\mu S_{r-1}(ve_{i-1},0)S_{r-1}(w_1,v_1)^T & \lambda\{S_{r-1}(v_1,w_1) + b_0 S_{r-1}(ve_{i-1},0)\} \\ -\mu S_{r-1}(w_1,v_1)^T & b_0 I_{2^{r-1}} \end{pmatrix} \\
&\times \begin{pmatrix} I_{2^{r-1}} & 0 \\ -\mu S_{r-1}(0,ve_{i-1})^T & I_{2^{r-1}} \end{pmatrix} \\
&= \begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{pmatrix}
\end{aligned}$$

where

$$\begin{aligned}
\alpha_{11} &= a_0 I_{2^{r-1}} - \lambda\mu\{S_{r-1}(ve_{i-1},0)S_{r-1}(w_1,v_1)^T + S_{r-1}(v_1,w_1)S_{r-1}(0,ve_{i-1})^T\} \\
&\quad -\lambda\mu b_0 S_{r-1}(ve_{i-1},0)S_{r-1}(0,ve_{i-1})^T \\
&= (a_0 - \lambda\mu v b_{i-1})I_{2^{r-1}}, \\
\alpha_{12} &= \lambda\big(S_{r-1}(v_1,w_1) + b_0 S_{r-1}(ve_{i-1},0)\big) = \lambda S_{r-1}(v_1 + b_0 ve_{i-1}, w_1), \\
\alpha_{21} &= -\mu(S_{r-1}(w_1,v_1)^T + b_0 S_{r-1}(0,ve_{i-1})^T) = -\mu S_{r-1}(w_1, v_1 + b_0 ve_{i-1})^T, \\
\alpha_{22} &= b_0 I_{2^{r-1}}.
\end{aligned}$$

Thus,

$$\begin{aligned}
\begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{pmatrix} &= \begin{pmatrix} (a_0 - \lambda\mu v b_{i-1})I_{2^{r-1}} & \lambda S_{r-1}(v_1 + b_0 ve_{i-1}, w_1) \\ -\mu S_{r-1}(w_1, v_1 + b_0 ve_{i-1})^T & b_0 I_{2^{r-1}} \end{pmatrix} \\
&= S_r^{\lambda,\mu}(v - \lambda\mu v b_{i-1}e_1 + b_0 ve_i, w).
\end{aligned}$$

This proves (i).

We have by Lemma 2.2,

$E_r{}^{\lambda,\mu}(e_i^*)(v)^{top}S_r^{\lambda,\mu}(v,w)E_r{}^{\lambda,\mu}(e_i^*)(v)^{bot}$

$$= \begin{pmatrix} a_0 I_{2^{r-1}} - \lambda\mu S_{r-1}(0, \nu e_{i-1}) S_{r-1}(w_1, v_1)^T & \lambda\{S_{r-1}(v_1, w_1) + b_0 S_{r-1}(0, \nu e_{i-1})\} \\ -\mu S_{r-1}(w_1, v_1)^T & b_0 I_{2^{r-1}} \end{pmatrix}$$
$$\times \begin{pmatrix} I_{2^{r-1}} & 0 \\ -\mu S_{r-1}(\nu e_{i-1}, 0)^T & I_{2^{r-1}} \end{pmatrix}$$
$$= \begin{pmatrix} \beta_{11} & \beta_{12} \\ \beta_{21} & \beta_{22} \end{pmatrix}$$

where

$$\begin{aligned} \beta_{11} &= a_0 I_{2^{r-1}} - \lambda\mu\{S_{r-1}(0, \nu e_{i-1}) S_{r-1}(w_1, v_1)^T + S_{r-1}(v_1, w_1) S_{r-1}(\nu e_{i-1}, 0)^T\} \\ &\quad -\lambda\mu b_0 S_{r-1}(0, \nu e_{i-1}) S_{r-1}(\nu e_{i-1}, 0)^T \\ &= (a_0 - \lambda\mu\nu a_{i-1}) I_{2^{r-1}}, \\ \beta_{12} &= \lambda\big(S_{r-1}(v_1, w_1) + b_0 S_{r-1}(0, \nu e_{i-1})\big) = \lambda S_{r-1}(v_1, w_1 + b_0 \nu e_{i-1}), \\ \beta_{21} &= -\mu(S_{r-1}(w_1, v_1)^T + b_0 S_{r-1}(\nu e_{i-1}, 0)^T) = -\mu S_{r-1}(w_1 + b_0 \nu e_{i-1}, v_1)^T, \\ \beta_{22} &= b_0 I_{2^{r-1}}. \end{aligned}$$

Thus,

$$\begin{aligned} \begin{pmatrix} \beta_{11} & \beta_{12} \\ \beta_{21} & \beta_{22} \end{pmatrix} &= \begin{pmatrix} (a_0 - \lambda\mu\nu a_{i-1}) I_{2^{r-1}} & \lambda S_{r-1}(v_1, w_1 + b_0 \nu e_{i-1}) \\ -\mu S_{r-1}(w_1 + b_0 \nu e_{i-1}, v_1)^T & b_0 I_{2^{r-1}} \end{pmatrix} \\ &= S_r^{\lambda,\mu}(v - \lambda\mu\nu a_{i-1} e_1, w + b_0 \nu e_i) \\ &= S_r^{\lambda,\mu}\big(vE_{i1}(-\lambda\mu\nu), wE_{1i}(\nu)\big). \end{aligned}$$

This proves (ii).

$${E_r}^{\lambda,\mu}(e_i)(\nu)^{bot} S_r^{\lambda,\mu}(v, w) {E_r}^{\lambda,\mu}(e_i)(\nu)^{top}$$
$$= \begin{pmatrix} a_0 I_{2^{r-1}} & \lambda S_{r-1}(v_1, w_1) \\ -\mu\{a_0 S_{r-1}(0, \nu e_{i-1})^T + S_{r-1}(w_1, v_1)^T\} & b_0 I_{2^{r-1}} - \lambda\mu S_{r-1}(0, \nu e_{i-1})^T S_{r-1}(v_1, w_1) \end{pmatrix}$$
$$\times \begin{pmatrix} I_{2^{r-1}} & \lambda S_{r-1}(\nu e_{i-1}, 0) \\ 0 & I_{2^{r-1}} \end{pmatrix}$$
$$= \begin{pmatrix} \gamma_{11} & \gamma_{12} \\ \gamma_{21} & \gamma_{22} \end{pmatrix}$$

where

$$\begin{aligned} \gamma_{11} &= a_0 I_{2^{r-1}}, \\ \gamma_{12} &= \lambda\big(S_{r-1}(v_1, w_1) + a_0 S_{r-1}(\nu e_{i-1}, 0)\big) = \lambda S_{r-1}(v_1 + a_0 \nu e_{i-1}, w_1), \\ \gamma_{21} &= -\mu(S_{r-1}(w_1, v_1)^T + a_0 S_{r-1}(0, \nu e_{i-1})^T) = -\mu S_{r-1}(w_1, v_1 + a_0 \nu e_{i-1})^T, \\ \gamma_{22} &= b_0 I_{2^{r-1}} - a_0 \lambda\mu S_{r-1}(0, \nu e_{i-1})^T S_{r-1}(\nu e_{i-1}, 0) \\ &\quad -\lambda\mu\{S_{r-1}(0, \nu e_{i-1})^T S_{r-1}(v_1, w_1) + S_{r-1}(w_1, v_1)^T S_{r-1}(\nu e_{i-1}, 0)\} \\ &= (b_0 - \lambda\mu\nu b_{i-1}) I_{2^{r-1}}. \end{aligned}$$

Thus

$$\begin{pmatrix}\gamma_{11} & \gamma_{12}\\ \gamma_{21} & \gamma_{22}\end{pmatrix} = \begin{pmatrix} a_0 I_{2^{r-1}} & \lambda S_{r-1}(v_1 + a_0 \nu e_{i-1}, w_1)\\ -\mu S_{r-1}(w_1, v_1 + a_0 \nu e_{i-1})^T & (b_0 - \lambda\mu\nu b_{i-1}) I_{2^{r-1}}\end{pmatrix}$$
$$= S_r^{\lambda,\mu}(v + a_0 \nu e_i, w - \lambda\mu\nu b_{i-1} e_1)$$
$$= S_r^{\lambda,\mu}\big(vE_{1i}(\nu), wE_{i1}(-\lambda\mu\nu)\big).$$

as required and (iii) follows.

$${E_r}^{\lambda,\mu}(e_i^*)(\nu)^{bot} S_r^{\lambda,\mu}(v,w) {E_r}^{\lambda,\mu}(e_i^*)(\nu)^{top}$$
$$= \begin{pmatrix} a_0 I_{2^{r-1}} & \lambda S_{r-1}(v_1, w_1)\\ -\mu\{a_0 S_{r-1}(\nu e_{i-1}, 0)^T + S_{r-1}(w_1, v_1)^T\} & b_0 I_{2^{r-1}} - \lambda\mu S_{r-1}(\nu e_{i-1}, 0)^T S_{r-1}(v_1, w_1)\end{pmatrix}$$
$$\times \begin{pmatrix} I_{2^{r-1}} & \lambda S_{r-1}(0, \nu e_{i-1})\\ 0 & I_{2^{r-1}}\end{pmatrix}$$
$$= \begin{pmatrix}\delta_{11} & \delta_{12}\\ \delta_{21} & \delta_{22}\end{pmatrix}$$

where

$$\begin{aligned}
\delta_{11} &= a_0 I_{2^{r-1}},\\
\delta_{12} &= \lambda\big(S_{r-1}(v_1, w_1) + a_0 S_{r-1}(0, \nu e_{i-1})\big) = \lambda S_{r-1}(v_1, w_1 + a_0 \nu e_{i-1}),\\
\delta_{21} &= -\mu(S_{r-1}(w_1, v_1)^T + a_0 S_{r-1}(\nu e_{i-1}, 0)^T)) = -\mu S_{r-1}(w_1 + a_0 \nu e_{i-1}, v_1)^T,\\
\delta_{22} &= b_0 I_{2^{r-1}} - a_0 \lambda\mu S_{r-1}(\nu e_{i-1}, 0)^T S_{r-1}(0, \nu e_{i-1})\\
&\quad -\lambda\mu\{S_{r-1}(\nu e_{i-1}, 0)^T S_{r-1}(v_1, w_1) + S_{r-1}(w_1, v_1)^T S_{r-1}(0, \nu e_{i-1})\}\\
&= (b_0 - \lambda\mu\nu a_{i-1}) I_{2^{r-1}}.
\end{aligned}$$

Hence,

$$\begin{pmatrix}\delta_{11} & \delta_{12}\\ \delta_{21} & \delta_{22}\end{pmatrix} = \begin{pmatrix} a_0 I_{2^{r-1}} & \lambda S_{r-1}(v_1, w_1 + a_0 \nu e_{i-1})\\ -\mu S_{r-1}(w_1 + a_0 \nu e_{i-1}, v_1)^T & (b_0 - \lambda\mu\nu a_{i-1}) I_{2^{r-1}}\end{pmatrix}$$
$$= {S_r}^{\lambda,\mu}(v, w - \lambda\mu\nu a_{i-1} e_1 + a_0 \nu e_i)$$

as required and (iv) follows. □

6. Conclusion

We have proved the Key Lemma which describes the action of the Almost Special Unimodular Vector group using the top bottom matrices. We will be using the Key Lemma to prove a Fundamental Property for Almost Suslin matrices, parallel to the one proved for Suslin Matrices. This will lead us to the equivalence of the Key Lemma and the Fundamental Property.

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